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A RELATION BETWEEN HOMOTOPY AND PIVOTAL METHODS FOR LINEAR COMPLEMENTARITY PROBLEMS

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A RELATION BETWEEN HOMOTOPY AND PIVOTAL METHODS

FOR LINEAR COMPLEMENTARITY PROBLEMS*

by

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RESUME

En utilisant des équations de transformation dûes à Mangasarian, on montre que les chemins de presque complémentarité sur certains polytopes convexes, peuvent être interprétés comme des chemins d'homotopie d'une fonction non linéaire, en choisissant convenablement le point initial de la fonction d'homotopie. Pour une classe spéciale de matrices, où le point initial est facilement identifié, on présente un algorithme de presque complémentarité par pivotage. L'utilisation d'une variable artificielle crée un chemin d'homotopie qui est précisément le chemin suivi par la méthode de Lemke.

ABSTRACT

Using transformation equations of Mangasarian, it is shown that almost complementarity paths on certain convex polytopes can be interpreted as a homotopy path of a nonlinear function. For a special class of matrices, for which the initial point is easily identified, an almost complementary pivoting algorithm is presented. The use of an artificial variable creates a homotopy path which is precisely the path followed by Lemke's method.

I. INTRODUCTION

The complementary pivoting algorithm of Carlton Lemke for resolving, under certain circumstances, the linear complementarity problem, is a technique familiar to mathematical programmers. In recent years, it has been studied extensively and has been shown to converge (or to "process" the problem) in various special cases, i.e., for certain classes of matrices. For relevant results and related bibliography see references [5]-[15], [17], [24], [26], [28], [31]-[40], [42], [43].

Lemke's algorithm is a pivotal method which traces a particular path of edges on a convex polytope. Along quite different lines, homotopy methods (derived from methods of continuation) for solving systems of nonlinear equations have been under extensive study for the past three or four years. On homotopy and related work, see, for example, references [1]-[4], [16], [18]-[23], [25], [27], [29], [30], [41], [44], [45]. Homotopy methods are known to have robust convergence properties and are known for example to succeed in many instances when classical Newton or quasi-Newton methods fail. Convergence theorems for such methods tend to be "global" in nature and often, in contrast to Newton-type theorems, require the starting point to be sufficiently far from a solution [22], [27].
In this paper a fundamental result of Mangasarian is used to show how these different streams of current interest can be related. Observed relations are first exploited for a very special class of linear complementarity problems and then extended to the general LCP. Several of the existence results obtained herein appear

to be new, at least in form. However, the wealth of past work in this area makes it clear that such results should be taken as highly tentative with regard to the question of whether or not they can be subsumed by what is already known.

The paper is organized as follows. In Section II a relation between homotopy and almost complementary paths is established and a new almost complementary pivoting algorithm is given for matrices containing a column whose off-diagonal elements all have the same nonzero sign. Lemke's method for matrices with a positive column is subsumed by the method described herein and an example is given for which the latter succeeds but Lemke's method with an artificial variable terminates without discovering the solution. In Section III existence results are given for this class of matrices (at least one column whose off-diagonal elements have the same nonzero sign). Theorems 1, 2 and 3 describe some members of the class Q (a Q-matrix is a matrix M such that the linear complementarity problem has a solution for any q). Theorem 4 gives a general existence result for matrices having an off-diagonal constant signed column and, for such matrices, Corollary 2 describes some members of the class Q. In Section IV the development is related to Lemke's method using an artificial variable for arbitrary matrices M. Theorems 5 and 6 give additional results on Q matrices.

II. A RELATION BETWEEN HOMOTOPY AND ALMOST COMPLEMENTARY PATHS

Consider the Linear Complementarity Problem:

(LCP) Find $z \in R^n \ni$

$$z \geq 0, \quad w = q + Mz \geq 0, \quad \langle z,w \rangle = 0$$

where q is a specified n-vector, M is a real $n \times n$ matrix, and $\langle .,. \rangle$ denotes the Euclidean inner product. The problem throughout will be assumed to be nondegenerate in the sense that every solution to $w = q + Mz$ has at least n nonzero components. The importance of (LCP) stems in part from the fact that it is a canonical form for some other problems in engineering plasticity, optimization theory, in game theory, fixed point problems, and for economic equilibrium conditions (not necessarily derived from an underlying optimization model--for example, in spatial equilibrium models). The above problem (LCP) is a special case of the nonlinear complementarity problem:

(NLCP) Find $z \in R^n \ni$

$$z \geq 0, \quad F(z) \geq 0, \quad \langle z, F(z) \rangle = 0$$

where $F: R^n \to R^n$ is a specified function. This is a more general canonical form (e.g., first-order optimality conditions for a general nonlinear program fall into this form) and is equivalent to the problem of variational inequalities in R^n. If one defines $F(z) = q + Mz$ then the problem (NLCP) obviously reduces to (LCP).

It was shown in [34] by Mangasarian that the problem (NLCP) can be converted to an equivalent problem of solving nonlinear equations. (For another interesting approach see [35], [36], [37].) Following Mangasarian, one first defines, for example, the function $G: R^n \to R^n$ as follows:[1]

$$G_i(z) = z_i^3 + F_i(z)^3 - |F_i(z) - z_i|^3, \quad i = 1, \ldots, n . \tag{1}$$

Then from the form of G it is easy to see that:

<u>Lemma 1</u>. For each i,

(i) $G_i(z) = 0 \Leftrightarrow z_i \geq 0, \; F_i(z) \geq 0, \; z_i F_i(z) = 0.$

(ii) $G_i(z) > 0 \Leftrightarrow z_i > 0$ and $F_i(z) > 0.$

(iii) $G_i(z) < 0 \Leftrightarrow z_i < 0$ or $F_i(z) < 0.$

The Mangasarian conclusion is immediate from (i): z is a solution to (NLCP) $\Leftrightarrow G(z) = 0$. Another important fact is that if $F(z)$ is C^2 then so is $G(z)$.

The discovery of Mangasarian opened up the prospect of solving complementarity problems (nonlinear or linear) with methods used for nonlinear equations. Along this line, an interesting possibility is to investigate whether or not the powerful homotopy tool may shed any new light on our knowledge, in particular about (LCP). This was discussed in an interesting paper by Watson [44] but the approach herein is quite different and leads to different results in terms of the path obtained and the existence of solutions.

For the following development any homotopy function could be considered. For specificity the Newton homotopy (associated with what is sometimes called the "global Newton method") will be employed. Let Q be

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[1] one could also use the simpler map G defined by $G_i(z) = \min\{z_i, F_i(z)\}$. This is the simplest function of Mangasarian's class. Lemma I applies to this G as well.

a specified C^2 function where $Q: R^n \to R^n$. Let $S = \{x \in R^n: \det Q'(x) = 0\}$. Assume

(A1) S is a set of measure zero in R^n.

(A2) $Q(x) = 0 \Rightarrow x \notin S$.

The second assumption (A2) is not generally required for convergence proofs but is desirable from the computational point of view to guarantee a finite path length. Now define the homotopy function

$$H(x,t,a): R^n \times R \times R^n - S \to R^n$$

as

$$H(x,t,a) = Q(x) - (1 - t)Q(a) .$$

Then by a parametrized version of Sard's Theorem [4] it follows that, for almost every $a \in R^n$, $\det Q'(a) \neq 0$ and the function $H_a: R^n \times R \to R^n$ given by

$$H_a(x,t) = Q(x) - (1 - t)Q(a) \tag{2}$$

is transversal to zero and hence $H_a^{-1}(0)$ is a C^1 one-dimensional manifold.

Let Γ_a denote the component of this manifold containing the point $(x,t) = (a,0)$. This component Γ_a, or often its projection into R^n, is loosely referred to as "the homotopy path." The homotopy method "follows" (by using simplicial approximation or differential equation techniques) the path Γ_a hopefully to some point $(x,1)$ for which it must be true, from (2), that $Q(x) = 0$ (since $H_a(x,1) = 0$).

Now, focusing on (LCP) replace $Q(z)$ by the C^2 function $G(z)$ specified by (1) where $F(z) = q + Mz = w$. Suppose that the initial point

a has the property that $G_i(a) = 0$ for $i \neq k$. Then for $i \neq k$,

$$H_a(z,t)_i = G_i(z) - (1 - t)G_i(a) = G_i(z) .$$

Hence,

$$H_a(z,t)_i = 0 \Leftrightarrow G_i(z) = 0$$

and for $i = k$,

$$H_a(z,t)_k = 0 \Leftrightarrow G_k(z) = (1 - t)G_k(a) .$$

But for the problem (LCP) we have from Lemma 1

$$G_i(z) = 0 \Leftrightarrow z_i \geq 0, \quad w_i \geq 0, \quad w_i z_i = 0 .$$

Thus on $H_a^{-1}(0)$ it is always true that

$$z_i \geq 0, \quad w_i \geq 0, \quad w_i z_i = 0, \qquad i \neq k.$$

This is the familiar k-almost complementary path of pivotal methods except that z_k and w_k are now unrestricted in sign. This k-ac path is the projection of the homotopy path Γ_a into R^n. The initial point a has been chosen in such a way that $H_a^{-1}(0)$ is now no longer C^1 (recall that Sard's Theorem only guarantees $H_a^{-1}(0)$ is a C^1 one-dimensional manifold for almost every $a \in R^n$).

However, the projection into R^n of Γ_a is a one-dimensional path with no crossings. The path is a path of adjacent complementary edges of the convex polytope

$$P_{-k} = \{z \in R^n \ni z_i \geq 0, \quad q_i + (Mz)_i \geq 0, \quad i \neq k\} .$$

Each corner on the path is an extreme point of this polytope. The fact that the path is not C^1 is not disturbing because it can be followed by pivoting.

The above described connection with homotopy suggests many questions and possible inquiries concerning, for example, transferability of known homotopy convergence theorems. In this paper only one specific line of inquiry is pursued.

At the outset it is observed that for a special class of matrices M there is an obvious way to find an initial point a as specified above. Suppose M has the property that all off-diagonal elements in the k^{th} column have the same nonzero sign. Choose the initial point a as follows:

$$a_i = 0,\ i \neq k,\ a_k = \lambda \operatorname{sgn} m_{ik},\ i \neq k \quad \text{for any} \quad \lambda >> 0 \ .$$

Thus if $m_{ik} < 0$ (> 0) $a_k << 0$ $(>> 0)$. This determines an initial point a such that $G_i(a) = 0$, $i \neq k$ (i.e., a lies on a k-ac ray of the polytope P_{-k}). For this special class of matrices the following algorithm can be described.

1. Basic variables are always z_k, w_k and one variable from each pair (z_i, w_i), $i \neq k$. The term "driving variable" will be used to denote the nonbasic variable being increased from the value zero. We shall say that z_k is the first driving variable, forced toward (and possibly through) the origin from the initial value a_k. The term "blocking variable" will be used to denote the first nonzero basic variable (+ or -) to become zero. The central idea is to keep $z_i \geq 0$, $w_i \geq 0$, and $z_i w_i = 0$, $i \neq k$. The variables z_k and w_k will always be unrestricted in sign.

2. Push the driving variable until the blocking variable is identified. If the blocking variable is $X_k \in \{z_k, w_k\}$ check the sign of $\overline{X}_k$

where $\bar{X}_k$ is the complement of X_k, i.e., $\{X_k, \bar{X}_k\} = \{z_k, w_k\}$. If $\bar{X}_k \geq 0$ then record a solution. Then, or if $\bar{X}_k < 0$, continue to push the driving variable until the blocking variable is neither z_k nor w_k. Its complement becomes the new driving variable which is now increased from zero. Each time an $X_k \in \{z_k, w_k\}$ becomes zero (blocking), check $\bar{X}_k$ for sign to see if a new solution has been found and continue as above. Termination always occurs on a ray, for the path cannot cycle.

Several remarks about the algorithm should be noted:

1. If the k^{th} column of M (<u>including</u> the diagonal term m_{kk}) is positive then the initial ray is feasible with regard to the inequalities $z \geq 0$, $q + Mz \geq 0$ and in this case the algorithm, until the point at which a first solution may be discovered is the same as Lemke's algorithm for matrices with a positive column.

2. The above procedure may find multiple solutions to (LCP).

3. The requirement that all off-diagonal elements have the same sign, for some column k, is very stringent. It will be removed in the final section of this paper.

4. One way to approach the existence of (LCP) solutions on the k-ac path is to observe the following:

a. If at least one of z_k and w_k are both initially negative ($a_k < 0$ and/or $(q + Ma)_k < 0$) and both z_k and w_k are positive on the terminal ray, then an odd number of solutions has been discovered.

b. If z_k and w_k are both initially positive and either (or both) is negative on the terminal ray then again an odd number of solutions has been found.

5. If the homotopy path Γ_a has a single component (i.e., if there is a single component to the set $G_i(z) = 0$, $i \neq k$) then the above algorithm must find all solutions to (LCP).

III. EXISTENCE RESULTS

An open question in the theory of linear complementarity problems is the following: What is the class of matrices M such that (LCP) has a solution for any q? This class is called Q-matrices. Although, because of the procedure described above for finding the initial point a, the above algorithm obviously applies to only a very limited class of matrices M, it is nevertheless true that at this point some information on the class Q can be given. The following lemma will be useful.

Lemma 2. Suppose M has a column k such that all off-diagonal terms are of the same nonzero sign. The k-ac algorithm terminates on a ray given by $\tilde{z} + \lambda h$, $\lambda \geq 0$, with $\tilde{z}$ denoting the last vertex of P_{-k} encountered and h a nonzero vector in R^n such that (i) $h_i \geq 0 \ \forall\, i \neq k$, (ii) $h_{i_0} > 0$ for some $i_0 \neq k$, (iii) $(Mh)_i \geq 0 \ \forall\, i \neq k$, (iv) $h_i(Mh)_i = 0 \ \forall\, i \neq k$.

Proof. At least one vertex will be encountered, since $m_{ik} \neq 0$ if $i \neq k$. The representation of the terminal ray as $\tilde{z} + \lambda h$, $\lambda \geq 0$, $h \neq 0$, and $\tilde{z}$ the last encountered extreme point of P_{-k} is definitional.

The fact that $h_i \geq 0$, $i \neq k$ follows because each z encountered on the k-ac path is "almost feasible" in the sense that $z_i \geq 0 \quad \forall\, i \neq k$. Hence $(z + \lambda h)_i \geq 0 \quad \forall\, i \neq k$ which implies $h_i \geq 0 \quad \forall\, i \neq k$. The existence of an $i_0 \neq k \ni h_{i_0} > 0$ is shown as follows. Since the algorithm is k-ac, it must be true that

$$\tilde{z}_i(q + M\tilde{z})_i = 0 \;\forall\, i \neq k,$$

and

$$\forall\, \lambda \geq 0, \quad (\tilde{z}_i + \lambda h_i)(q_i + (M(\tilde{z} + \lambda h))_i) = 0 \;\forall\, i \neq k\ .$$

Hence if $h_i = 0 \;\forall\, i \neq k$ then $h_k \neq 0$ (since $h \neq 0$) and $\tilde{z}_i(Mh)_i = 0$, which implies $\tilde{z}_i\left(\sum_{j \neq k} m_{ij}h_j + m_{ik}h_k\right) = 0 \;\forall\, i \neq k$. This implies that $\tilde{z}_i m_{ik} = 0 \;\forall\, i \neq k$ and hence that $\tilde{z}_i = 0 \;\forall\, i \neq k$. But this implies that the last extreme point encountered must be the same as the first. It is well known that this cannot happen on an almost complementary path which begins on a ray. The fact that $(Mh)_i \geq 0 \;\forall\, i \neq k$ follows from the fact that on the path of the algorithm it is always true that $w_i \geq 0 \;\forall\, i \neq k$ and hence, $\forall\, \lambda \geq 0$, $(q + M(\tilde{z} + \lambda h))_i \geq 0 \;\forall\, i \neq k$, which can be true only if $(Mh)_i \geq 0 \;\forall\, i \neq k$. The last asserted property, that

$$h_i(Mh)_i = 0 \;\forall\, i \neq k$$

is derived from

$$\forall\, \lambda \geq 0 \quad (\tilde{z}_i + \lambda h_i)(q_i + (M(\tilde{z} + \lambda h))_i) = 0 \;\forall\, i \neq k,$$

for $h_i > 0$ implies $\tilde{z}_i + \lambda h_i > 0$ which implies $q_i + (M(\tilde{z} + \lambda h))_i = 0 \;\forall\, \lambda \geq 0$, which can be true only if $(Mh)_i = 0$.

□

The following three simple and known theorems are presented to illustrate the framework just developed. Let $N = \{1, \ldots, n\}$.

Theorem 1. Let $J \subseteq N$ index the positive columns of M. Suppose $J \neq \phi$ and suppose the rows indexed by $N - J$ are also positive. Then for any q the k-ac path contains an odd number of solutions (and, hence, M is a Q-matrix).

Proof. Choose $k \in J$. Since the k^{th} column of M is positive the first ray is feasible ($a_k \gg 0$ and $w_k = (q + Ma)_k > 0$), in accord with the above remark 4(c). It suffices to show that $h_k < 0$ on the terminal ray. Suppose to the contrary that $h_k \geq 0$. By Lemma 2 there is a $j_0 \neq k$ such that $h_{j_0} > 0$. Then, for $i \in I = N - J$,

$$(Mh)_i = \sum_{\substack{j \neq j_0 \\ j \neq k}} \overset{+}{m_{ij}} \overset{\geq 0}{h_j} + \overset{+}{m_{ij_0}} \overset{+}{h_{j_0}} + \overset{+}{m_{ik}} \overset{\geq 0}{h_k} > 0 .$$

Hence, for $i \in I$, $h_i = 0$, which implies $j_0 \in J$. Thus,

$$(Mh)_{j_0} = \sum_{t \in I} m_{j_0 t} h_t + \sum_{t \in J - j_0 - k} \overset{+}{m_{j_0 t}} \overset{\geq 0}{h_t} + \overset{+}{m_{j_0 j_0}} \overset{+}{h_{j_0}} + \overset{+}{m_{j_0 k}} \overset{\geq 0}{h_k} > 0 .$$

Which implies $h_{j_0} (Mh)_{j_0} > 0$. This contradicts Lemma 2. $\square$

The next two theorems show additional members of the class Q.

Theorem 2. Suppose M has positive diagonal terms, the k^{th} column of M has negative off-diagonal terms, and all other entries in M are nonnegative. Then for any q the k-ac path contains an odd number of solutions (and, hence, M is a Q-matrix).

Proof. Since the first ray is infeasible it will suffice to show that $h_k > 0$ and $(Mh)_k > 0$ on the terminal ray. Suppose $h_k \leq 0$. By Lemma 2 there is an $i_0 \neq k$ such that $h_{i_0} > 0$. Then:

$$(Mh)_{i_0} = \sum_{\substack{j \neq k \\ j \neq i_0}} \overset{\geq 0}{m_{i_0 j}} \overset{\geq 0}{h_j} + \overset{-}{m_{i_0 k}} \overset{\leq 0}{h_k} + \overset{+}{m_{i_0 i_0}} \overset{+}{h_{i_0}} > 0 ,$$

which implies $h_{i_0}(Mh)_{i_0} > 0$, a contradiction to Lemma 2. Hence $h_k > 0$. Now:

$$(Mh)_k = \sum_{j \neq k} \overset{\geq 0}{m_{kj}} \overset{\geq 0}{h_j} + \overset{+}{m_{kk}} \overset{+}{h_k} > 0 . \qquad \square$$

Theorem 3. Suppose M has a positive column k. Let $M_{-k,-k}$ denote the matrix M with k^{th} column and k^{th} row deleted. Let $\tilde{M}$ denote a principal submatrix of $M_{-k,-k}$ and suppose that for any such $\tilde{M}$ the system

$$\tilde{M}\tilde{z} \leq 0, \quad \tilde{z} > 0$$

has no solution. Then for any q the k-ac path contains an odd number of solutions (and, hence, M is a Q-matrix).

Proof. Letting h be the vector defining the terminal ray, define $J = \{i \neq k : h_i > 0\}$. Since $h_i(Mh)_i = 0$, $i \neq k$, we have

$$h_i\left(\sum_{j \neq k} m_{ij} h_j + m_{ik} h_k\right) = 0, \quad i \neq k$$

which implies

$$\sum_{j \in J} m_{ij} h_j + m_{ik} h_k = 0, \quad i \in J .$$

If $h_k \geq 0$ then $\sum_{j \in J} m_{ij}h_j \leq 0$, $i \in J$, which contradicts the hypotheses of the theorem. Hence $h_k < 0$ and the theorem is proved.

□

In the next theorem existence of solutions is related to the signs of certain minors of M. The following notation will be employed. The determinant of any square matrix, say B, will be denoted as $|B|$ and without confusion the same notation $|J|$ will be used to denote the cardinality of a finite set J. Given the n-square matrix M and naturally ordered index sets $R \subseteq \{1, \ldots, n\}$, $S \subseteq \{1, \ldots, n\}$, with $|R| = |S|$, the notation $M_{R,S}$ will be used for the submatrix whose elements are m_{ij}, $i \in R$, $j \in S$.

Thus for example, $M_{R,R}$ is a principal submatrix and $|M_{R,R}|$ is a principal minor. The symbol J_k will be used to denote any naturally ordered subset of $(1, \ldots, n)$ containing the index k. If $J_k = (i_1, \ldots, i_p) \subseteq \{1, \ldots, n\}$, with $i_1 < i_2 \ldots < i_p$, and $s \in J_k$, then C_s denotes the ordinal position of s in J_k. That is, $i_{C_s} = s$.

<u>Theorem 4</u>. Suppose all off-diagonal elements in the k^{th} column of M have the same nonzero sign. Let J_k be any naturally ordered subset of $(1, \ldots, n)$ such that $k \in J_k$ and $|J_k| \geq 2$. For any such set J_k suppose there is a $t \in J_k - \{k\}$ such that $|M_{J_k-\{k\},J_k-\{t\}}| \neq 0$. Suppose also that:

A. If $m_{ik} < 0$, $i \neq k$ or $q_k + \lambda m_{kk} < 0$ $(\lambda >> 0)$ then $\forall\ J_k$

$$(-1)^{(C_k+C_t)} \frac{|M_{J_k-\{k\},J_k-\{k\}}|}{|M_{J_k-\{k\},J_k-\{t\}}|} > 0 \quad \text{and} \quad (-1)^{(C_k+C_t)} \frac{|M_{J_k,J_k}|}{|M_{J_k-\{k\},J_k-\{t\}}|} > 0 \ .$$

B. If $m_{ik} > 0$ and $q_k + \lambda m_{kk} \geq 0$ $(\lambda >> 0)$ then

$$(-1)^{(C_k+C_t)} \frac{|M_{J_k-\{k\},J_k-\{k\}}|}{|M_{J_k-\{k\},J_k-\{t\}}|} \leq 0 \quad \text{or} \quad (-1)^{(C_k+C_t)} \frac{|M_{J_k,J_k}|}{|M_{J_k-\{k\},J_k-\{t\}}|} < 0$$

for each J_k. Then the k-ac path contains an odd number of solutions.

<u>Proof</u>. By Lemma 2, the algorithm must terminate on a ray given by $\tilde{z} + \lambda h$, $\lambda \geq 0$ for some point $\tilde{z} \in R^n$ and a vector $h \in R^n$ with $h_i \geq 0$, $i \neq k$, and for at least one $i \neq k$, $h_i > 0$, and $h_i(Mh)_i = 0 \; \forall \; i \neq k$. Let $p \leq n - 1$ be such that $h_{i_1}, \ldots, h_{i_p}$ denotes the set of h_i which are positive, $i \neq k$, and let J_k denote the natural ordering of the indices $\{i_1, \ldots, i_p\} \cup \{k\}$.

Then using the fact that

$$\sum_{j \in J_k} m_{ij}h_j = 0, \qquad i \in J_k - \{k\} ,$$

or

$$\sum_{j \in J_k - \{t\}} m_{ij}h_j = -m_{it}h_t, \quad i \in J_k - \{k\} ,$$

Cramer's rule can be used to compute

$$h_k = (-1)^{(C_k+C_t)} \frac{|M_{J_k-\{k\},J_k-\{k\}}|}{|M_{J_k-\{k\},J_k-\{t\}}|} h_t \tag{3}$$

$$h_j = (-1)^{(C_j+C_t)} \frac{|M_{J_k-\{k\},J_k-\{j\}}|}{|M_{J_k-\{k\},J_k-\{t\}}|} h_t, \quad j \in J_k - \{t\} - \{k\} . \tag{4}$$

Now, using (3) and (4), $(Mh)_k$ is computed as follows:

$$(Mh)_k = \sum_{j\in J_k} m_{kj}h_j = m_{kt}h_t + \sum_{j\in J_k-\{t\}} (-1)^{(C_j+C_t)} \frac{\left|M_{J_k-\{k\},J_k-\{j\}}\right|}{\left|M_{J_k-\{k\},J_k-\{t\}}\right|} h_t$$

$$= \frac{\left[m_{kt}\left|M_{J_k-\{k\},J_k-\{t\}}\right| + \sum_{j\in J_k-\{t\}} (-1)^{(C_j-C_t)}\left|M_{J_k-\{k\},J_k-\{j\}}\right|\right]h_t}{\left|M_{J_k-\{k\},J_k-\{t\}}\right|}$$

$$= (-1)^{(C_k+C_t)} \frac{\left|M_{J_k,J_k}\right|}{\left|M_{J_k-\{k\},J_k-\{t\}}\right|} \quad . \tag{5}$$

Since $t \in J_k - \{k\}$ it is true by definition of J_k that $h_t > 0$. Under condition A the initial ray is such that $z_k > 0$ and $w_k < 0$, or $z_k < 0$, $w_k \geq 0$, or $z_k < 0$ and $w_k < 0$, i.e., the initial ray is infeasible with regard to the inequalities $z \geq 0$, $w = q + Mz \geq 0$. Using (3) and (5), condition A guarantees that $z_k > 0$ and $w_k > 0$ on the terminal ray, which implies that an odd number of solutions has been encountered. Under condition B the initial ray is feasible ($z_k > 0$ and $w_k \geq 0$). Again using (3) and (5), condition B guarantees that $z_k < 0$ or $w_k < 0$ on the terminal ray, which again implies that an odd number of solutions has been encountered.

□

In the above theorem the terms $\left|M_{J_k,J_k}\right|$ and $\left|M_{J_k-\{k\},J_k-\{k\}}\right|$ are $2^n - 2$ in number and consist of all principal minors of M with the exception of m_{kk}. The terms $\left|M_{J_k-\{k\},J_k-\{t\}}\right|$ are $2^{n-1} - 1$ special minors of M.

The above theorem, when interpreted in R^2, has the following form.

<u>Corollary 1</u>. In R^2, suppose M has a nonzero off-diagonal element, m_{ik}. Suppose

1. If $m_{ik} < 0$ or $q_k + \lambda m_{kk} < 0$ $(\lambda >> 0)$ then

$$\frac{m_{ii}}{m_{ik}} < 0 \quad \text{and} \quad \frac{|M|}{m_{ik}} < 0 \ .$$

2. If $m_{ik} > 0$ and $q_k + \lambda m_{kk} \geq 0$ $(\lambda >> 0)$ then

$$\frac{m_{ii}}{m_{ik}} > 0 \quad \text{or} \quad \frac{|M|}{m_{ik}} > 0 \ .$$

Then the k-ac path contains an odd number of solutions.

The assumptions of Corollary 1 apply to the problem defined by

$$M = \begin{bmatrix} -1 & +1 \\ +2 & -1 \end{bmatrix} \qquad q = \begin{bmatrix} -1 \\ 4 \end{bmatrix} .$$

Lemke's method (with an artificial variable) does not succeed in finding the solution $(z_1 = 0, z_2 = 4)$ to this problem. However, taking k to be 1 or 2, Condition 1 of the corollary is satisfied. □

The hypotheses of Theorem 4 are independent of q if $m_{kk} \neq 0$. Thus we obtain the result:

<u>Corollary 2</u>. Suppose the hypotheses of Theorem 4 are satisfied with $m_{kk} \neq 0$. Then for any q the k-ac path contains an odd number of solutions and hence M is a Q-matrix. □

As examples, the above matrix $\begin{bmatrix} -1 & +1 \\ +2 & -1 \end{bmatrix}$ is an element of Q. Another example is the matrix $\begin{bmatrix} 0 & + \\ - & + \end{bmatrix}$, taking $k = 2$. This latter example shows that nonzero principal minors are not required by Theorem 4, for in this case the minor $m_{ii} = m_{11}$ is zero.

The positive diagonal matrices, which are members of Q, are not covered by Corollary 2. These will be added by the development in the next section.

IV. RELATION TO LEMKE'S METHOD WITH ARTIFICIAL VARIABLE

The assumption is now dropped that M has a column with all off-diagonal elements of the same nonzero sign. In this case there may not exist a point a such that, for some k, $G_i(a) = 0$, $i \neq k$ (with G given by (1)). An example is $w_1 = -1 - z_1$, $w_2 = -1 - z_2$. Moreover, even if a point a exists $\ni G_i(a) = 0$, $i \neq k$, there may be no k-ac ray ρ for which $G_i(z) = 0 \;\forall\, i \neq k$, $\forall\, z \in \rho$. The convergence arguments of Section 3 require the existence of an initial k-ac ray. Finally, even if there is such a ray it may not be easy to find. The approach of Lemke is one way to overcome this difficulty. Another way is given in Section V.

The approach of Lemke is to transform the original problem (LCP) to the new augmented problem:

Find $z \in R^n$, $z_{n+1} \in R \ni$

(LCP)'
$$W = q + Mz + \alpha z_{n+1} \geq 0$$

$$W_{n+1} = q_{n+1} - \sum_{i=1}^{n} z_i \geq 0$$

$$z \geq 0,\; z_{n+1} \geq 0,\quad \langle W,z\rangle = 0,\quad W_{n+1} z_{n+1} = 0$$

where q_{n+1} is a very large positive number and $\alpha \in R^n$ is, for the moment, any n-vector such that $\alpha_i > 0$, $i = 1, \ldots, n$. We choose the initial point a as $a_i = 0$, $i = 1, \ldots, n$, $a_{n+1} >> 0$ and follow an $n+1$-ac path for this problem. Thus on the first ray, initially, $z_i = 0$, $i = 1, \ldots, n$, $z_{n+1} >> 0$, $w_i > 0$, $i = 1, \ldots, n$, $w_{n+1} > 0$. Hence the first ray is feasible for (LCP)'. Clearly q_{n+1} can be set sufficiently large that $h_{n+1} < 0$ will imply z_{n+1} passes through the value 0 an odd number of times before w_{n+1} changes sign. Thus an odd number of solutions to (LCP)' with $z_{n+1} = 0$ will be guaranteed, which in turn guarantees that the $n+1$-ac path has found an odd number of solutions to the original problem (LCP).

The algorithm for (LCP)' is the same as Lemke's algorithm up to the point of finding a first solution $(z_{n+1} = 0)$ if in fact a first solution is found. The algorithm thus far, as with Lemke's method, requires only that $\alpha > 0$. However, the main existence result to be presented, Theorem 5, postulates additional conditions on α. The approach to be followed will parallel the approach taken in the previous section. It will be assumed that the elements of the matrix for (LCP)' are arranged in the form:

$$\begin{bmatrix} \bar{M} \\ -1, \ldots, -1, 0 \end{bmatrix} = \begin{bmatrix} m_{11} & \cdots & m_{1n} & \alpha_1 \\ \vdots & & \vdots & \vdots \\ m_{n1} & \cdots & m_{nn} & \alpha_n \\ -1 & \cdots & -1 & 0 \end{bmatrix}.$$

However, since w_{n+1} will always remain positive (until "far out" on the terminal ray), there is no need explicitly to consider the last equation or to worry about the value of q_{n+1}. Now, following the results of the previous section, associated with the terminal ray is a distinguished nonzero vector h with components $h_1, \ldots, h_n, h_{n+1}$ such that $h_i \geq 0$, $i = 1, \ldots, n$, $h_{j_0} > 0$ for some $j_0 \in \{1, \ldots, n\}$ and such that

$(\bar{M}h)_i = \sum_{j=1}^{n} m_{ij}h_j + \alpha_i h_{n+1} \geq 0$, $i = 1, \ldots, n$, and $h_i(\bar{M}h)_i = 0$, $i = 1, \ldots, n$. Before presenting Theorem 5 several known results on Q-matrices can be easily observed.

Lemma 3. If M is a P matrix (positive principal minors) then $h_{n+1} < 0$ for any q, and hence P is a Q-matrix.

Proof. For $i = 1, \ldots, n$,

$$h_i(\bar{M}h)_i = 0 \Leftrightarrow h_i\left(\sum_{j=1}^{n} m_{ij}h_j + \alpha_i h_{n+1}\right) = 0 \ .$$

Since $h_i \geq 0$, $i = 1, \ldots, n$, it follows that $h_{n+1} > 0$ implies $h_i\left(\sum_{j=1}^{n} m_{ij}h_j\right) \leq 0$. It is well known that if M is a P matrix the conditions $h_i \geq 0$ and $h_i(Mh)_i < 0$, $i = 1, \ldots, n$ imply that $h_i = 0$, $i = 1, \ldots, n$. But this cannot be since $h_{j_0} > 0$ for some $j_0 \in \{1, \ldots, n\}$. This contradiction implies that $h_{n+1} < 0$. □

From Lemma 3, if M is a P matrix then for any q the $n+1$-ac path contains an odd number of solutions to (LCP). (It is known that when M is a P matrix the solution is unique for any q.)

Lemma 4. Suppose $M \geq 0$ and M has positive diagonal elements. Then for any q the $n+1$-ac path contains an odd number of solutions, and hence M is a Q-matrix.

Proof. Suppose $h_{n+1} \geq 0$ and let $i_0 \in \{1, \ldots, n\}$ be such that $h_{i_0} > 0$. Then

$$(\bar{M}h)_{i_0} = \sum_{\substack{j \neq i_0 \\ j \neq n+1}} \overset{\geq 0}{m_{i_0 j}} \overset{\geq 0}{h_j} + \overset{+}{m_{i_0 i_0}} \overset{+}{h_{i_0}} + \overset{+}{\alpha_{i_0}} \overset{\geq 0}{h_{n+1}} > 0$$

and we have the contradiction $h_{i_0}(\overline{M}h)_{i_0} > 0$. □

Lemma 5. Suppose M has the property that for every principal submatrix $\tilde{M}$ the system

$$\tilde{M}\tilde{z} \leq 0, \quad \tilde{z} > 0$$

has no solution. Then for any q the $n+1$-ac path contains an odd number of solutions and, hence, M is a Q-matrix.

Proof. Again suppose $h_{n+1} \geq 0$. Let J index the positive components of h_i, $i = 1, \ldots, n$, and let $\tilde{z}$ be the subvector consisting of these positive h_i. Then

$$h_i(\overline{M}h)_i = 0, \quad i = 1, \ldots, n \Rightarrow \tilde{z}_i\left(\sum_{j \in J} m_{ij}\tilde{z}_j + \alpha_i h_{n+1}\right) = 0, \quad i \in J$$

which implies $\sum\limits_{j \in J} m_{ij}\tilde{z}_j \leq 0$ for $i \in J$. Thus we have $\tilde{z} > 0$, $M_{J,J}\tilde{z} \leq 0$, a contradiction. □

It is known that the class of matrices hypothesized in Lemma 5 includes strictly semimonotone matrices, which in turn includes P matrices n as well as all square nonnegative matrices with positive diagonal elements [8]. Thus Lemma 5 subsumes Lemmas 3 and 4. Lemmas 3 through 5 require only that the n-vector α be positive. The following existence result covers situations which may require a more judicious choice of the positive vector.

Given a nonempty ordered subset $J \subset \{1, \ldots, n\}$, with $t \in J$, let $J - \{t\} + \{n + 1\}$ denote the ordered index set obtained by deleting from J the index t and adding in the last position the index $n + 1$ denoting the last column of $\bar{M}$ (recall that the vector α appears in the last column of $\bar{M}$). The symbol C_t denotes the ordinal position of the index t in the ordered set J.

Theorem 5. Suppose:

1. M has nonzero principal minors.

2. There is a positive vector $\alpha \in R^n$ such that for any principal minor, say $M_{J,J}$, where J is some nonempty naturally ordered subset of $\{1, \ldots, n\}$, there is a $t \in J$ such that the minor

$$\left|M_{J,J-\{t\}+\{n+1\}}\right| \neq 0 \quad \text{and} \quad (-1)^{|J|+C_t+1} \frac{|M_{J,J}|}{M_{J,J-\{t\}+\{n+1\}}} < 0 .$$

Then for any q the $n+1$-ac path contains an odd number of solutions and, hence, M is a Q-matrix.

Proof. The theorem follows from Theorem 4 when k is taken to be $n + 1$, $J_{n+1} - \{n + 1\}$ is replaced with $J \subset \{1, \ldots, n\}$ and it is recognized that $C_{n+1} = |J| + 1$. □

This result classifies members of the class Q according to (i) the existence of nonzero principal minors and (ii) the existence of a positive n-vector α which produces a negative sign for quotients involving the principal minors and other specified minors. It can be verified that the positive diagonal matrices satisfy the hypotheses of Theorem 5. As a special case we can put Theorems 4 and 5 together, in R^2, to obtain the following result.

<u>Theorem 6</u>. Suppose $M \in R^{2\times 2}$ satisfies one of the following:

1. M has a column k such that $m_{ik} \neq 0$, $m_{kk} \neq 0$ $(i \neq k)$ and such that

   a. $m_{ik} > 0$, $m_{kk} > 0 \Rightarrow m_{ii} > 0$ or $|M| > 0$.

   b. $m_{ik} < 0$ or $m_{kk} < 0 \Rightarrow \dfrac{m_{ii}}{m_{ik}} < 0$ and $\dfrac{|M|}{m_{ik}} < 0$.

2. M is a positive diagonal matrix.

Then M is a Q-matrix.

It can be shown that Theorem 6 actually characterizes the Q-matrices in R^2.

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